

# MINIMAX PROPERTIES OF BETA KERNEL DENSITY ESTIMATORS

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*Abstract:* In this paper, we are interested in the study of beta kernel estimators from an asymptotic minimax point of view. It is well known that beta kernel estimators are—on the contrary of classical kernel estimators—“free of boundary effect” and thus are very useful in practice. The goal of this paper is to prove that there is a price to pay: for very regular functions or for certain losses, these estimators are not minimax. Nevertheless they are minimax for classical regularities such as regularity of order two or less than two, supposed commonly in the practice and for some classical losses.

*Key words and phrases:* Beta Kernel, Density, Minimax estimation.

## 1. Introduction

This paper is devoted to the study of some properties concerning beta kernel estimators.

These estimators were introduced by Chen (1999) for density estimation with support in  $[0, 1]$ . Indeed, contrary to classical kernel estimators, they are (in the classical case of the estimation of a twice differentiable density) “free of boundary effect”: their bias tends to 0 (even at points 0 and 1) and their mean integrated square error is of order  $n^{-4/5}$ .

This property has contributed to popularize their use in many applied fields such as economy and finance. Number of papers deal with these applications. Among others let us point out Bouezmarni and Rollin (2003), Bouezmarni and van Bellegem (2009) and Charpentier and Oulidi (2010).

In this article, we adopt a quite different point of view in order to study the performance of these beta kernel estimators. We put ourselves in an asymptotic minimax framework. In particular, our study will not be restricted to twice differentiable density or to mean integrated square error.

We study here the performance of beta kernel estimators in the density model for densities belonging to Hölder spaces. We observe  $n$  independent and identi-

cally distributed (i.i.d.) variables  $X_1, \dots, X_n$  which admit the unknown density  $f$  with respect to the Lebesgue measure on  $[0, 1]$ . Moreover, we assume that  $f$  belongs to the class of functions  $\Sigma(\beta, L)$  where  $\beta > 0$  is a regularity parameter and  $L > 0$  is a Lipschitz constant which are assumed to be known by the statistician. The class  $\Sigma(\beta, L)$  is the set of all the density functions defined on  $[0, 1]$  which are  $m$  times differentiable such that for all  $(x, y) \in [0, 1]^2$ :

$$|f^{(m)}(x) - f^{(m)}(y)| \leq L|x - y|^{\beta-m},$$

where  $m = \sup\{\ell \in \mathbb{N} : \ell < \beta\}$ .

We measure the quality of the estimators of the unknown density function  $f$  with a risk in  $L^p$  loss. More precisely, if  $\tilde{f}_n$  is an arbitrary estimator, we define its risk over  $\Sigma(\beta, L)$ , for  $p \geq 1$ , by

$$R_n(\tilde{f}_n, \beta, L) = \sup_{f \in \Sigma(\beta, L)} R_n(\tilde{f}_n, f)$$

where

$$R_n(\tilde{f}_n, f) = \left( E_f^n \left( \|\tilde{f}_n - f\|_p^p \right) \right)^{\frac{1}{p}},$$

and  $E_f^n$  is the expectation with respect to the law of  $(X_1, \dots, X_n)$ . The minimax rate of convergence on  $\Sigma(\beta, L)$  is defined as  $r_n(\beta, L) = \inf_{\tilde{f}_n} R_n(\tilde{f}_n, \beta, L)$  where the infimum is taken over all the estimators. The asymptotic of  $r_n(\beta, L)$  is well-known up to a constant (see Ibragimov and Hasminskii (1981)) and is of order  $\varphi_n(\beta) = n^{-\frac{\beta}{2\beta+1}}$ . We are then interested in knowing if beta kernel estimators are optimal estimators in minimax framework, in other words, if they converge at the rate  $\varphi_n(\beta)$ .

The advantage of this approach with respect to the classical one (second order regularity and mean integrated square error) is that our study is quite precise concerning the understanding of the expression “free of boundary effect”. Our first result, Theorem 1, illustrates that, for regularities less or equal than 2 and for  $L^p$  losses with  $p < 4$ , it is possible to construct an optimal beta kernel estimator. Of course, this is linked to the fact that, even on the boundary of  $[0, 1]$ , the bias term tends to 0. Nevertheless, our second result, Theorem 2, shows that for higher regularities ( $\beta > 2$ ), even if this bias term tends to 0, the order of convergence is not good. This leads to the impossibility to construct an optimal beta kernel estimator in that case. To conclude our study, we show, in

Theorem 3, that even if the bias term is of the good order (for  $\beta \leq 2$ ), for  $L^p$  losses such that  $p \geq 4$  the variance term is not good. This leads again to the impossibility to construct an optimal beta kernel estimator.

Finally, let us point out that our results clarify the conditions for using beta kernel estimator by showing some intrinsic limitations of these estimators. Nevertheless they have good properties of convergence for small regularities as it is often supposed for practical purposes. This work is a first step in the study of beta kernel estimators from an asymptotic minimax point of view. There is still a lot to do in order to complete this study such as finding data-driven methods to chose the bandwidth of these estimators (cross validation methods or Lepski type procedures).

In Section 2, we introduce our estimators and give our main results. Section 3 is devoted to the proofs.

## 2. Main results

Before stating our main results, let us recall the definition of beta kernel estimators.

**Definition 1.** *For all  $b \in (0, 1)$  and  $t \in [0, 1]$ , let us introduce the following density:*

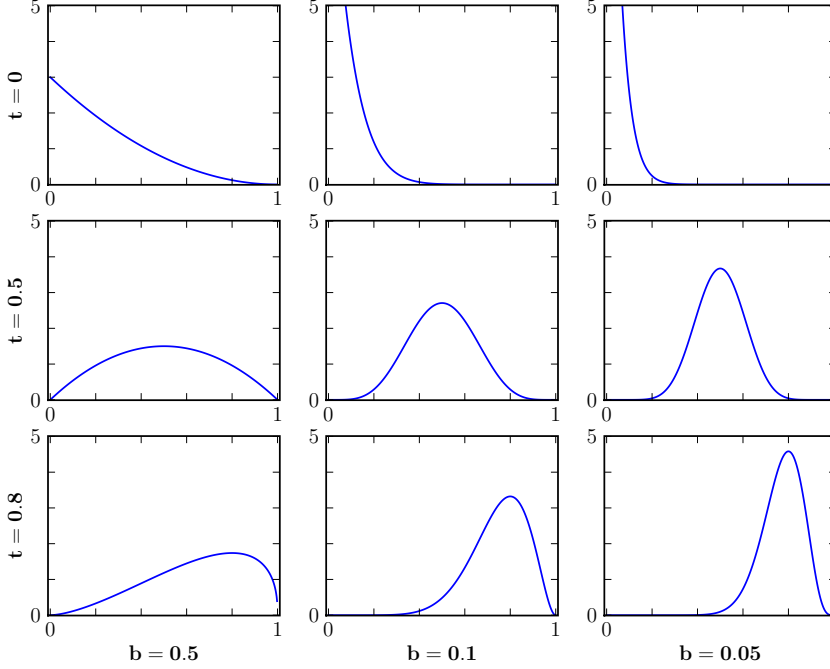
$$K_{t,b}(x) = \frac{x^{\frac{t}{b}}(1-x)^{\frac{1-t}{b}}}{B\left(\frac{t}{b} + 1, \frac{1-t}{b} + 1\right)} I_{[0,1]}(x) \quad (2.1)$$

where  $B(\cdot, \cdot)$  is the standard beta function and  $I$  denotes the characteristic function. Following Chen (1999) let us introduce the associated beta kernel estimator:

$$\hat{f}_b(t) = \frac{1}{n} \sum_{k=1}^n K_{t,b}(X_k), \quad t \in [0, 1].$$

**Remark 1.** *Let us notice that  $(\hat{f}_b)_{b \in (0,1)}$  defines a one-parameter family of estimators. Note also that the density given by (2.1) corresponds to a beta distribution of parameters  $t/b + 1$  and  $(1-t)/b + 1$ .*

Figure 2.1 represents some beta kernels  $K_{t,b}$  drawn for different values of  $t$  and  $b$ . From left to right,  $b$ —which plays the role of a bandwidth—decreases and one can observe a concentration of the kernel in a neighborhood of  $t$  which is the mode.

Figure 2.1: Some beta kernels for different values of  $b$  and  $t$ 

We will give three results about beta kernel estimators. The first one is a positive result: for  $0 < \beta \leq 2$  and  $1 \leq p < 4$  there exists a beta kernel estimator with properly chosen *bandwidth* which achieves the minimax rate of convergence on  $\Sigma(\beta, L)$  in  $L^p$ -loss.

**Theorem 1.** *Set  $1 \leq p < 4$  and  $0 < \beta \leq 2$ . Set  $b_n = cn^{-\frac{2}{2\beta+1}}$ , where  $c$  is a positive constant. Then the estimator  $\hat{f}_{b_n}$  achieves the rate  $r_n(\beta, L)$ . More precisely:*

$$\limsup_{n \rightarrow \infty} \frac{R_n(\hat{f}_{b_n}, \beta, L)}{r_n(\beta, L)} < \infty.$$

The following theorem shows that, for regularities larger than 2 and most of all losses, beta kernel density estimators are not minimax.

**Theorem 2.** *Set  $p \geq 2$  and  $\beta > 2$ . Then the family of estimators  $\hat{f}_{b_n}$  satisfies,*

for all sequence  $(b_n)_{n \in \mathbb{N}}$  in  $(0, 1)$ ,

$$\liminf_{n \rightarrow \infty} \frac{R_n(\hat{f}_{b_n}, \beta, L)}{r_n(\beta, L)} = \infty.$$

The next theorem proves, for regularities less or equal than 2, contrary to what happens in Theorem 1, that the beta kernel density estimator is not minimax if the risk is measured with  $L_p$  losses with  $p \geq 4$ .

**Theorem 3.** *Set  $p \geq 4$  and  $0 < \beta \leq 2$ . Then the family of estimators  $\hat{f}_{b_n}$  satisfies, for all sequence  $(b_n)_{n \in \mathbb{N}}$  in  $(0, 1)$ ,*

$$\liminf_{n \rightarrow \infty} \frac{R_n(\hat{f}_{b_n}, \beta, L)}{r_n(\beta, L)} = \infty.$$

In the framework of Theorem 1, the bias term of beta kernel estimators is of order  $b^{\beta/2}$  and variance term  $(nb^{1/2})^{-1/2}$ . These orders are the same as for classical kernel estimators with  $h = b^{1/2}$ . A trade off between bias and variance leads to choose  $b$  of the form  $cn^{-\frac{2}{2\beta+1}}$  and the associated beta kernel estimator converges to  $f$  at rate  $\varphi_n(\beta)$ . Let us remark that this theorem can be viewed as a generalization of the “classical case” which corresponds to  $\beta = 2$  and  $p = 2$ .

When  $\beta > 2$  and  $p \geq 2$ , the impossibility to construct an optimal beta kernel estimator is linked with the construction of two functions which belong to  $\Sigma(\beta, L)$ . The first one has its bias term that cannot go to 0 faster than the rate  $b$  while the variance term of the second one is lower bounded by  $(nb^{1/2})^{-1/2}$ . This implies that beta kernel estimators cannot converge to  $f$  at a rate faster than  $\varphi_n(2)$  in this framework.

In the case of Theorem 3, the non-optimality of beta kernel estimators is linked with the construction of a function, which belongs to  $\Sigma(\beta, L)$ , such its variance term is not of good order (by at least an extra  $|\log b|$  factor). We exhibit a second function in  $\Sigma(\beta, L)$  with a bias term of order  $b^{\beta/2}$  that allows us to conclude that beta kernel estimators cannot converge at the minimax rate  $\varphi_n(\beta)$ .

Proofs of these three theorems are given in Section 3.

### 3. Proofs

In all the proofs,  $C$  denotes a positive constant that can change of values from line to line. In the following,  $(a_n) \asymp (b_n)$  stands for  $0 < \liminf_{n \rightarrow \infty} a_n/b_n \leq \limsup_{n \rightarrow \infty} a_n/b_n < +\infty$ .

### 3.1 Proof of Theorem 1

Firstly, set  $1 \leq p < 4$ ,  $0 < \beta \leq 2$ ,  $L > 0$  and  $f \in \Sigma(\beta, L)$ . As the function  $f$  (and the number of observations  $n$ ) is always fixed in this proof, we will denote, for simplicity,  $E$  instead of  $E_f^n$ . We have:

$$\begin{aligned} R_n(\hat{f}_b, f) &= \left( E \left( \|\hat{f}_b - f\|_p^p \right) \right)^{1/p} \\ &= \left( E \left( \int_0^1 |\hat{f}_b(t) - f(t)|^p dt \right) \right)^{1/p} \\ &\leq C \left( \int_0^1 |B_t|^p dt + E \left( \int_0^1 |Z_t|^p dt \right) \right)^{1/p}, \end{aligned}$$

where  $B_t = E(\hat{f}_b(t)) - f(t)$  and  $Z_t = \hat{f}_b(t) - E(\hat{f}_b(t))$ . Thus, as  $p \geq 1$ , we obtain:

$$R_n(\hat{f}_b, f) \leq C \left\{ \left( \int_0^1 |B_t|^p dt \right)^{\frac{1}{p}} + \left( \int_0^1 E(|Z_t|^p) dt \right)^{\frac{1}{p}} \right\}. \quad (3.1)$$

The proof of our theorem will be derived from two lemmas. The first one is used to control the integrated bias term and the second one to control the integrated moment of the centered stochastic term  $Z_t$ .

**Lemma 1.** *Set  $p$  and  $\beta$  as in Theorem 1. Then, for all  $0 < b < 1$ :*

$$\left( \int_0^1 |B_t|^p dt \right)^{\frac{1}{p}} \leq C L b^{\frac{\beta}{2}}. \quad (3.2)$$

**Lemma 2.** *Set  $p$  and  $\beta$  as in Theorem 1. Then, for all  $0 < b < 1$ :*

$$\left( \int_0^1 E(|Z_t|^p) dt \right)^{\frac{1}{p}} \leq C \left( \frac{1}{n b^{\frac{1}{2}}} \right)^{\frac{1}{2}}.$$

Proofs of these lemmas are in Subsection 3.3 and 3.4. Let us complete the proof of Theorem 1. From Equation (3.1) and Lemmas 1 and 2 we derive:

$$R_n(\hat{f}_b, f) \leq C \left\{ L b^{\frac{\beta}{2}} + \left( \frac{1}{n b^{\frac{1}{2}}} \right)^{\frac{1}{2}} \right\}.$$

As this equation is valid for all  $0 < b < 1$  we deduce that, if  $b_n$  denotes the quantity  $c n^{-\frac{2}{2\beta+1}} \asymp \arg \min_{0 < b < 1} (L b^{\beta/2} + (n b^{1/2})^{-(1/2)})$  with  $c$  a positive constant, we have :

$$R_n(\hat{f}_{b_n}, f) \leq C \left\{ L b_n^{\frac{\beta}{2}} + \left( \frac{1}{n b_n^{\frac{1}{2}}} \right)^{\frac{1}{2}} \right\} \leq C r_n(\beta, L),$$

that gives the result of the theorem.

### 3.2 Proofs of Theorem 2 and 3

In all these proofs we will use auxiliary functions. Let us define the first one:

$$f_0(x) = I_{[0,1]}(x).$$

Next, for  $0 < \beta \leq 2$ , we define

$$f_\beta(x) = 1 + L_\beta \sum_{k=1}^{2N} (-1)^{k+1} \left( \frac{1}{(4N)^\beta} - \left| x - \frac{2k-1}{4N} \right|^\beta \right) I_{[\frac{k-1}{2N}, \frac{k}{2N}]}(x),$$

where  $N$  is the integer part of  $b^{-1/2}/20$  and  $L_\beta = \frac{L}{2} \min(1, 1/\beta)$ . Figure 3.2 shows the function  $f_\beta$  for two values of  $\beta$ . Finally we consider:

$$f_3(x) = 2xI_{[0,1]}(x).$$

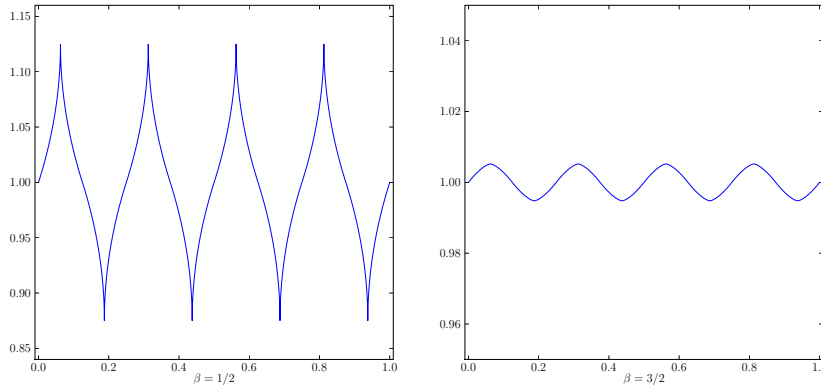


Figure 3.2: Function  $f_\beta$ :  $\beta = 1/2$  in the left side,  $\beta = 3/2$  in the right side. In both cases,  $L = 1$  and  $b = 0.0005$ .

For simplicity, we will use the notation  $E_\alpha$  instead of  $E_{f_\alpha}^n$ . These functions satisfy the following lemma:

**Lemma 3.** *For all  $p \geq 2$  and  $b \in (0, 1)$ , we have:*

*i) The function  $f_0$  belongs to  $\Sigma(\beta, L)$  for all  $\beta > 0$  and  $L > 0$  and satisfies:*

$$E_0 \left( \|\hat{f}_b - f_0\|_p^p \right) \geq C d_n(b, p) \left( n b^{1/2} \right)^{-p/2}, \quad (3.3)$$

where

$$d_n(b, p) = \int_{\frac{1}{2}}^{1-b} \left( \frac{1}{1-t} \right)^{\frac{p}{4}} dt.$$

ii) Set  $0 < \beta \leq 2$  and  $L > 0$ . The function  $f_\beta$  (which also depends on  $b$  and  $L$ ) belongs to  $\Sigma(\beta, L)$  and satisfies:

$$E_\beta \left( \|\hat{f}_b - f_\beta\|_p^p \right) \geq C b^{p\beta/2}, \quad (3.4)$$

iii) The function  $f_3$  belongs to  $\Sigma(\beta, L)$  for all  $\beta > 2$  and  $L > 0$  and satisfies:

$$E_3 \left( \|\hat{f}_b - f_3\|_p^p \right) \geq C b^p \quad (3.5)$$

Let  $\beta > 2$ ,  $L > 0$  and  $p \geq 2$ . We have for all  $b \in (0, 1)$ , using (3.3) and (3.5):

$$\begin{aligned} \sup_{f \in \Sigma(\beta, L)} R_n(\hat{f}_b, f) &\geq \frac{1}{2} \left( R_n(\hat{f}_b, f_0) + R_n(\hat{f}_b, f_3) \right) \\ &\geq C \inf_{b \in (0, 1)} \left( b + \left( \frac{1}{n b^{1/2}} \right)^{1/2} \right) \asymp \varphi_n(2). \end{aligned}$$

Since  $\varphi_n(\beta) = o(\varphi_n(2))$ , this implies Theorem 2.

Let  $0 < \beta \leq 2$ ,  $L > 0$  and  $p \geq 4$ . We have for all  $b \in (0, 1)$ :

$$\sup_{f \in \Sigma(\beta, L)} R_n(\hat{f}_b, f) \geq \frac{1}{2} \left( R_n(\hat{f}_b, f_0) + R_n(\hat{f}_b, f_\beta) \right).$$

Then we have, using (3.3) and (3.4):

$$\begin{aligned} \sup_{f \in \Sigma(\beta, L)} R_n(\hat{f}_b, f) &\geq C \left( b^{\beta/2} + \frac{d_n(b, p)}{(n b^{1/2})^{1/2}} \right) \\ &\geq C \left( b^{\beta/2} + \frac{|\log b|}{(n b^{1/2})^{1/2}} \right). \end{aligned}$$

The function  $f(b) = b^{\beta/2} + |\log b|/(n b^{1/2})^{-1/2}$  attains its minimum  $b_n^0$  on  $(0, 1)$  for  $n$  large enough. The minimum satisfies  $b_n^0 = c_1 n^{-2/(2\beta+1)} (c_2 + |\log(b_n^0)|)^{4/(2\beta+1)}$  with  $c_1$  and  $c_2$  positive constants. It can be easily proved that the sequence  $b_n^0$  tends to 0 as  $n$  tends to  $\infty$ . This implies that

$$\inf_{b \in (0, 1)} \left\{ b^{\beta/2} + \frac{|\log b|}{(n b^{1/2})^{1/2}} \right\} = (b_n^0)^{\beta/2} + \frac{|\log b_n^0|}{(n (b_n^0)^{1/2})^{1/2}} \asymp \varphi_n(\beta) |\log(b_n^0)|^{2\beta/(2\beta+1)}.$$

This last result implies Theorem 3.



### 3.3 Proof of lemma 1

To prove inequality (3.2), it is sufficient to prove that  $|B_t| \leq CLb^{\frac{\beta}{2}}$ , for all  $t \in [0, 1]$ . In order to prove this last inequality, we will use the following result (see Johnson, Kotz and Balakrishnan, 1994).

**Lemma 4.** *If  $\xi$  is a variable with density  $K_{t,b}$  ( $\xi \sim K_{t,b}$ ) then there exist two functions  $\Delta_1$  and  $\Delta_2$ , and a constant  $M > 0$  such that:*

$$E(\xi) - t = b(1 - 2t) + \Delta_1(t, b) \quad (3.6)$$

and

$$\sigma^2(\xi) = bt(1 - t) + \Delta_2(t, b), \quad (3.7)$$

where

$$\sup_{0 \leq t \leq 1} |\Delta_j(t, b)| \leq Mb^2 \quad \text{if } j = 1, 2. \quad (3.8)$$

The variable  $\xi$  introduced in this lemma appears in the study of the bias term. Indeed we have

$$\begin{aligned} B_t &= E(\hat{f}_b(t)) - f(t) \\ &= E\left(\frac{1}{n} \sum_{k=1}^n K_{t,b}(X_k)\right) \\ &= \int_0^1 K_{t,b}(x) f(x) dx - f(t) \\ &= E(f(\xi) - f(t)). \end{aligned}$$

Now, we have to distinguish two cases depending on the position of  $\beta$  with respect to 1.

**a.** Assume that  $0 < \beta \leq 1$ . Since  $f$  belongs to  $\Sigma(\beta, L)$ , one can write:

$$\begin{aligned} |B_t| &\leq E(|f(\xi) - f(t)|) \\ &\leq LE(|\xi - t|^\beta) \\ &\leq LE(|\xi - t|)^\beta. \end{aligned}$$

Last inequality holds because  $\beta \leq 1$ . Set  $\eta > 0$ .

$$\begin{aligned}
|B_t| &\leq LE \left( |\xi - t| \left( I_{|\xi - t| \leq \eta} + I_{|\xi - t| > \eta} \right) \right)^\beta \\
&\leq L \left( \eta + \sqrt{\sigma^2(\xi) \cdot P(|\xi - t| > \eta)} \right)^\beta \\
&\leq L \left( \eta + \sqrt{\sigma^2(\xi) \cdot \frac{\sigma^2(\xi)}{\eta^2}} \right)^\beta \\
&\leq L \left( \eta + \frac{Cb}{\eta} \right)^\beta.
\end{aligned}$$

Except for the last inequality which follows from Equation (3.7) and (3.8), all the others follow from classical probability inequalities. Optimizing in  $\eta$ , we obtain that  $|B_t| \leq CLb^{\frac{\beta}{2}}$ , which allows to conclude.

**b.** Assume that  $1 < \beta \leq 2$ . Using the mean-value theorem, we obtain:

$$\begin{aligned}
|B_t| &= |E(f(\xi) - f(t))| \\
&= \left| E \left( f'(t)(\xi - t) + (\xi - t)(f'(\tilde{\xi}) - f'(t)) \right) \right|,
\end{aligned}$$

where  $|\tilde{\xi} - t| \leq |\xi - t|$ . Thus we obtain:

$$\begin{aligned}
|B_t| &= \left| f'(t)(E(\xi) - t) + E \left( (\xi - t)(f'(\tilde{\xi}) - f'(t)) \right) \right| \\
&\leq |f'(t)| \cdot |E(\xi) - t| + E \left( |\xi - t| \cdot |f'(\tilde{\xi}) - f'(t)| \right) \\
&\leq A(M+1)b + LE \left( |\xi - t|^\beta \right),
\end{aligned}$$

where  $A = \sup_{f \in \Sigma(\beta, L)} \|f'\|_\infty < +\infty$  (see, for example, Tsybakov (2004), §1.2.1). Remark that last inequality follows from Equation (3.6).

Let us consider the term  $E \left( |\xi - t|^\beta \right)$ . By introducing the mean of  $\xi$  we obtain:

$$\begin{aligned}
E \left( |\xi - t|^\beta \right) &= E \left( |\xi - E(\xi) + E(\xi) - t|^\beta \right) \\
&\leq C \left( E \left( |\xi - E(\xi)|^\beta \right) + |E(\xi) - t|^\beta \right).
\end{aligned}$$

As  $1 < \beta \leq 2$ , we obtain, using Hölder's inequality:

$$\begin{aligned}
E \left( |\xi - t|^\beta \right) &\leq C \left( \sigma^\beta(\xi) + |E(\xi) - t|^\beta \right) \\
&\leq C \left( (Mb)^{\frac{\beta}{2}} + (Mb)^\beta \right) \\
&\leq Cb^{\frac{\beta}{2}}.
\end{aligned}$$

Finally, we obtain again that  $|B_t| \leq CLb^{\frac{\beta}{2}}$  that gives the lemma in the second case.

### 3.4 Proof of Lemma 2

Note that  $Z_t = \hat{f}_b(t) - E(\hat{f}_b(t)) = \frac{1}{n} \sum_{k=1}^n \eta_k$ , where the  $\eta_k = K_{t,b}(X_k) - E(K_{t,b}(X_k))$ 's are i.i.d. centered variables. Thanks to the following lemma (see Bretagnolle and Hubert, 1979) it is possible to control precisely the  $p$ -th moment of  $Z_t$ .

**Lemma 5.** *If  $y_1, \dots, y_n$  are  $n$  i.i.d. variables such that  $E(y_1) = 0$  and  $\sigma^2(y_1) = v$ , then the following inequalities hold:*

1. *If  $p \leq 2$  then:*

$$E \left( \left| \frac{1}{n} \sum_{k=1}^n y_k \right|^p \right) \leq \left( \frac{v}{n} \right)^{\frac{p}{2}}.$$

2. *If  $p > 2$  and moreover  $\|y_1\|_{\infty} < +\infty$ , then:*

$$E \left( \left| \frac{1}{n} \sum_{k=1}^n y_k \right|^p \right) \leq C_p \left( \frac{v \|y_1\|_{\infty}^{p-2}}{n^{p-1}} + \left( \frac{v}{n} \right)^{\frac{p}{2}} \right).$$

This lemma will be apply with  $y_k = \eta_k$ , ( $k = 1, \dots, n$ ). Thus we have to control two terms:  $\sigma^2(\eta_1)$  on the one hand and  $\|\eta_1\|_{\infty}$  on the other hand.

**a.** In order to control of  $\sigma^2(\eta_1)$ , let us compute:

$$\begin{aligned} \sigma^2(\eta_1) &\leq E \left( (K_{t,b}(X_1))^2 \right) \\ &= \int_0^1 \frac{x^{2\frac{t}{b}} (1-x)^{2\frac{1-t}{b}}}{B^2 \left( \frac{t}{b} + 1, \frac{1-t}{b} + 1 \right)} f(x) dx \\ &= A_b(t) E[f(Z)], \end{aligned}$$

where:

$$A_b(t) = \frac{B \left( 2\frac{t}{b} + 1, 2\frac{1-t}{b} + 1 \right)}{B^2 \left( \frac{t}{b} + 1, \frac{1-t}{b} + 1 \right)} \quad (3.9)$$

and  $Z$  has a beta distribution with parameters  $2t/b + 1$  and  $2(1-t)/b + 1$ . Since  $f$  belongs to  $\Sigma(\beta, L)$  and thanks to the fact that the supremum of  $\|f\|_{\infty}$

over  $\Sigma(\beta, L)$  is finite, we obtain that  $\sigma^2(\eta_1) \leq CA_b(t)$ . Moreover, it is known (see Chen, 2000) that for  $b$  small enough:

$$A_b(t) \leq \frac{Cb^{-\frac{1}{2}}}{\sqrt{t(1-t)}}.$$

Thus, for all  $b$  small enough:

$$\sigma^2(\eta_1) \leq \frac{Cb^{-\frac{1}{2}}}{\sqrt{t(1-t)}}.$$

**b.** Control of  $\|\eta_1\|_\infty$ . As  $t$  is the mode of  $K_{t,b}$  and thanks to Stirling's formula, it can be shown (see Chen, 2000) that:

$$\sup_{x \in [0,1]} K_{t,b}(x) \leq \frac{Cb^{-\frac{1}{2}}}{\sqrt{t(1-t)}}.$$

Thus we obtain:

$$\begin{aligned} \|\eta_1\|_\infty &= \|K_{t,b}(X_1) - E(K_{t,b}(X_1))\|_\infty \\ &\leq \|K_{t,b}(X_1)\|_\infty + |E(f(\xi))| \\ &\leq \sup_{x \in [0,1]} K_{t,b}(x) + |f(t)| + |E(f(\xi)) - f(t)| \\ &\leq \frac{Cb^{-\frac{1}{2}}}{\sqrt{t(1-t)}} + \sup_{f \in \Sigma(\beta, L)} \|f\|_\infty + CLb^{\frac{\beta}{2}}. \end{aligned}$$

Thus, for small  $b$  we have:

$$\|\eta_1\|_\infty \leq \frac{Cb^{-\frac{1}{2}}}{\sqrt{t(1-t)}}.$$

Now let us complete the proof of this lemma. We have to distinguish two cases. The first one concerns the case where  $p \leq 2$ . The second one, the case where  $p > 2$ .

First, let us assume that  $p \leq 2$ . Applying Lemma 5 and using the bound on  $\sigma^2(\eta_1)$  just obtained, we have:

$$\begin{aligned} E(|Z_t|^p) &= E\left(\left|\frac{1}{n} \sum_{k=1}^n \eta_k\right|^p\right) \\ &\leq \left(\frac{\sigma^2(\eta_1)}{n}\right)^{\frac{p}{2}} \\ &\leq C \left(\frac{1}{nb^{\frac{1}{2}}}\right)^{\frac{p}{2}} (t(1-t))^{-\frac{p}{4}}. \end{aligned}$$

Last, if  $p > 2$ , we have, thanks to Lemma 5:

$$\begin{aligned} E(|Z_t|^p) &= E\left(\left|\frac{1}{n} \sum_{k=1}^n \eta_k\right|^p\right) \\ &\leq C_p \left\{ \frac{\sigma^2(\eta_1) \|\eta_1\|_\infty^{p-2}}{n^{p-1}} + \left(\frac{\sigma^2(\eta_1)}{n}\right)^{\frac{p}{2}} \right\}. \end{aligned}$$

Thanks to our bound on  $\|\eta_1\|_\infty$  we obtain:

$$\begin{aligned} E(|Z_t|^p) &\leq C_p \left\{ \frac{\sigma^2(\eta_1) \sigma^{2p-4}(\eta_1)}{n^{p-1}} + \left(\frac{\sigma^2(\eta_1)}{n}\right)^{\frac{p}{2}} \right\} \\ &\leq C_p \left\{ \left(\frac{\sigma^2(\eta_1)}{n}\right)^{p-1} + \left(\frac{\sigma^2(\eta_1)}{n}\right)^{\frac{p}{2}} \right\} \\ &\leq C_p \left\{ \left(\frac{\sigma^2(\eta_1)}{n}\right)^{p-1-p/2} + 1 \right\} \left(\frac{\sigma^2(\eta_1)}{n}\right)^{\frac{p}{2}} \\ &\leq C \left(\frac{\sigma^2(\eta_1)}{n}\right)^{\frac{p}{2}}. \end{aligned}$$

Using our bound on  $\sigma^2(\eta_1)$  we obtain:

$$E(|Z_t|^p) \leq C \left(\frac{1}{nb^{\frac{1}{2}}}\right)^{\frac{p}{2}} (t(1-t))^{-\frac{p}{4}}.$$

Taking all together lemma follows. Indeed, for any  $p < 4$  and  $b$  small enough we have:

$$\left(\int_0^1 E(|Z_t|^p) dt\right)^{\frac{1}{p}} \leq C \left(\frac{1}{nb^{\frac{1}{2}}}\right)^{\frac{1}{2}} \left(\int_0^1 (t(1-t))^{-\frac{p}{4}} dt\right)^{\frac{1}{p}}.$$

### 3.5 Proof of Lemma 3

Let us consider a preliminary lemma which will be proved in Subsection 3.6.

**Lemma 6.** *For all  $p \geq 2$  and  $b \in (0, 1)$  and all density  $f$ , we have:*

$$E\left(\|\hat{f}_b - f\|_p^p\right) \geq \int_0^1 |B_t|^p dt \quad (3.10)$$

and:

$$E\left(\|\hat{f}_b - f\|_p^p\right) \geq 2^{-p} \int_0^1 \left(E\left(Z_t^2\right)\right)^{\frac{p}{2}} dt. \quad (3.11)$$

*Proof of i).* In order to prove (3.3) it is enough, thanks to (3.11), to lower bound  $\int_0^1 (E_0(Z_t^2))^{\frac{p}{2}} dt$ .

Note that  $E_0(Z_t^2)$  can be written in the following way:

$$\begin{aligned} E_0(Z_t^2) &= \frac{1}{n} \left( E_0(\zeta_t^2) - (E_0(\zeta_t))^2 \right) \\ &= \frac{1}{n} \left( E_0(\zeta_t^2) - 1 \right) \end{aligned}$$

where  $\zeta_t = K_{t,b}(X_1)$ . Thus our goal is to minorate  $E_0(\zeta_t^2)$ . We have:

$$E_0(\zeta_t^2) = E(K_{t,b}^2(X_1)) = \int_0^1 K_{t,b}^2(x) dx = A_b(t)$$

where  $A_b(t)$  is defined by Equation (3.9).

Let us introduce the  $R$ -function defined as follows, for  $z \geq 0$ :

$$R(z) = \frac{1}{\Gamma(z+1)} \left( \frac{z}{e} \right)^z \sqrt{2\pi z}.$$

It is well-known that  $R$  is an increasing function such that  $R(z) < 1$  and,  $R(z) \rightarrow 1$  as  $z \rightarrow +\infty$ .

Following Chen (1999) let us write  $A_b(t)$  in terms of  $R$ -functions:

$$A_b(t) = c(b) \frac{1}{\sqrt{t(1-t)}} \frac{R^2(\frac{t}{b}) R^2(\frac{1-t}{b}) R(\frac{2}{t} + 1)}{R(\frac{2t}{b}) R(\frac{2(1-t)}{b}) R^2(\frac{1}{b} + 1)}$$

where

$$c(b) = \frac{e^{-1}}{2\sqrt{\pi}} b \left( 1 + \frac{1}{b} \right)^{\frac{3}{2}} \left( 1 + \left( \frac{2}{b} + \frac{5}{2} \right)^{-1} \right)^{\frac{2}{b}+3} \geq Cb^{-\frac{1}{2}},$$

for  $b$  small enough. Moreover, as  $R$  is increasing we have:

$$\frac{R^2(\frac{t}{b}) R^2(\frac{1-t}{b}) R(\frac{2}{b} + 1)}{R(\frac{2t}{b}) R(\frac{2(1-t)}{b}) R^2(\frac{1}{b} + 1)} \geq \frac{R^2(\frac{t}{b}) R^2(\frac{1-t}{b})}{R(\frac{2t}{b}) R(\frac{2(1-t)}{b}) R(\frac{1}{b} + 1)}.$$

Using the fact that  $R(z) < 1$ , we obtain:

$$\frac{R^2(\frac{t}{b}) R^2(\frac{1-t}{b}) R(\frac{2}{b} + 1)}{R(\frac{2t}{b}) R(\frac{2(1-t)}{b}) R^2(\frac{1}{b} + 1)} \geq R^2\left(\frac{t}{b}\right) R^2\left(\frac{1-t}{b}\right).$$

Thus we have:

$$\begin{aligned}
A_b(t) &\geq C \frac{b^{-\frac{1}{2}}}{\sqrt{t(1-t)}} R^2\left(\frac{t}{b}\right) R^2\left(\frac{1-t}{b}\right) \\
&\geq C \frac{b^{-\frac{1}{2}}}{\sqrt{t(1-t)}} R^2\left(\frac{t}{b}\right) R^2\left(\frac{1-t}{b}\right) I_{[1/2, 1-b]}(t) \\
&\geq C \frac{b^{-\frac{1}{2}}}{\sqrt{t(1-t)}} R^2\left(\frac{1}{2b}\right) R^2(1) I_{[1/2, 1-b]}(t).
\end{aligned}$$

For  $b$  small enough,  $R^2\left(\frac{1}{2b}\right)$  is greater than  $R^2(1)$ . Hence we obtain:

$$A_b(t) \geq C \frac{b^{-\frac{1}{2}}}{\sqrt{t(1-t)}}. \quad (3.12)$$

From Equation (3.12) we obtain:

$$\begin{aligned}
E_0\left(\zeta_t^2\right) &\geq C b^{-\frac{1}{2}} \frac{1}{\sqrt{t(1-t)}} I_{[1/2, 1-b]}(t) \\
&\geq C b^{-\frac{1}{2}} \frac{1}{\sqrt{1-t}} I_{[1/2, 1-b]}(t).
\end{aligned}$$

Thus, for  $b$  small enough we obtain:

$$E_0\left(Z_t^2\right) \geq C \frac{1}{n b^{1/2}} \frac{1}{\sqrt{1-t}} I_{[1/2, 1-b]}(t) - \frac{1}{n}.$$

Finally, we can write:

$$\int_0^1 \left(E_0\left(Z_t^2\right)\right)^{\frac{p}{2}} dt \geq C \left(\frac{1}{n b^{1/2}}\right)^{\frac{p}{2}} d_n(b, p).$$

*Proof of ii).* We define for  $k = 1 \dots, N$ ,  $t_k = (2k-1)/(4N)$  and the following intervals:  $T_k = [t_k - \varepsilon b^{1/2}, t_k + \varepsilon b^{1/2}]$ ,  $I_k = [t_k - b^{1/2}, t_k + b^{1/2}]$  and  $J_k = [t_k - 1/(4N), t_k + 1/(4N)]$  where  $0 < \varepsilon < 1/2$  will be chosen. Let us recall that  $1/(4N)$  is very close to  $5b^{1/2}$ . Note that we have the following inclusions:  $T_k \subset I_k \subset J_k$ .

Now, let us lower bound the integrated bias:

$$\begin{aligned}
\int_0^1 |B_t| dt &\geq \sum_{\ell=1}^N \int_{T_{2\ell-1}} \left| \int_0^1 K_{t,b}(x) (f_\beta(t) - f_\beta(x)) dx \right| dt \\
&\geq \sum_{\ell=1}^N \int_{T_{2\ell-1}} (A_\ell(t) - B(t)) dt
\end{aligned}$$

where

$$A_\ell(t) = \int_{J_{2\ell-1} \cap I_{2\ell-1}^c} K_{t,b}(x) (f_\beta(t) - f_\beta(x)) dx$$

and

$$B(t) = \int_{\{f_\beta(x) \geq f_\beta(t)\}} K_{t,b}(x) (f_\beta(x) - f_\beta(t)) dx.$$

Since for  $t \in T_{2\ell-1}$  and  $x \in J_{2\ell-1} \cap I_{2\ell-1}^c$  we have:

$$\begin{aligned} f_\beta(t) - f_\beta(x) &= L_\beta \left( |x - t_\ell|^\beta - |t - t_\ell|^\beta \right) \\ &\geq L_\beta \left( b^{\beta/2} - \varepsilon^\beta b^{\beta/2} \right) \\ &\geq C b^{\beta/2}, \end{aligned}$$

it follows that:

$$\begin{aligned} A_\ell(t) &\geq C b^{\beta/2} \int_{J_\ell \cap I_\ell^c} K_{t,b}(x) dx \\ &= C b^{\beta/2} P(\xi \in J_{2\ell-1} \cap I_{2\ell-1}^c), \end{aligned}$$

where  $\xi \sim K_{t,b}$ . This probability can be estimated:

$$\begin{aligned} P(\xi \in J_{2\ell-1} \cap I_{2\ell-1}^c) &\geq P(t_{2\ell-1} + b^{1/2} < \xi < t_{2\ell-1} + 5b^{1/2}) \\ &\geq P(t_{2\ell-1} - t + t + b^{1/2} < \xi < t_{2\ell-1} - t + t + 5b^{1/2}) \\ &\geq P(t + 2b^{1/2} < \xi < t + 4b^{1/2}) \\ &\geq 2b^{1/2} K_{t,b}(t + 4b^{1/2}) \end{aligned}$$

For  $1/4 \leq t \leq 3/4$  we have:

$$K_{t,b}(t + 4b^{1/2}) \underset{b \rightarrow 0}{\sim} K_{t,b}(t) \exp\left(-\frac{8}{t(1-t)}\right) \geq C b^{-1/2}.$$

Indeed, using the  $R$  function we obtain:

$$\begin{aligned} K_{t,b}(t) &= \frac{R\left(\frac{t}{b}\right) R\left(\frac{1-t}{b}\right) \sqrt{b}(1+1/b)}{R(1/b) \sqrt{2\pi t(1-t)}} \\ &\geq C b^{-1/2} \end{aligned}$$

for  $1/4 \leq t \leq 3/4$  and  $b$  small enough. Thus, for  $\ell$  such that  $T_{2\ell-1} \subset [1/4, 3/4]$ ,  $t \in T_{2\ell-1}$  and for  $b$  small enough, we have:

$$A_\ell(t) \geq C b^{\beta/2},$$



where  $C$  is an absolute constant independent of  $b$  and  $\varepsilon$ .

For  $t \in T_{2\ell-1}$  we have:

$$\begin{aligned} B(t) &= \int_{\bigcup_{k=1}^N \{|x-t_{2k-1}| \leq |t-t_{2\ell-1}|\}} K_{t,b}(x) (f_\beta(x) - f_\beta(t)) dx \\ &\leq \sum_{k=1}^N \int_{\{|x-t_{2k-1}| \leq |t-t_{2\ell-1}|\}} K_{t,b}(x) L_\beta(|t-t_{2\ell-1}|^\beta - |x-t_{2k-1}|^\beta) dx \\ &\leq C\varepsilon^\beta b^{\beta/2}. \end{aligned}$$

Taking all together we obtain the following lower bound:

$$\begin{aligned} \int_0^1 |B_t| dt &\geq \sum_{\substack{\ell=1 \\ T_{2\ell-1} \subset [1/4, 3/4]}}^N \int_{T_{2\ell-1}} A_\ell(t) dt - \sum_{\ell=1}^N \int_{T_{2\ell-1}} B(t) dt \\ &\geq CN\varepsilon b^{1/2} b^{\beta/2} - N\varepsilon^{\beta+1} b^{1/2} b^{\beta/2} \\ &\geq b^{\beta/2} (c_1\varepsilon - c_2\varepsilon^{\beta+1}) \end{aligned}$$

where  $c_1$  and  $c_2$  are absolute positive constants. Thus, for  $\varepsilon = \min(1/2, (c_1/c_2)^{1/\beta}/2)$  we obtain:

$$\int_0^1 |B_t| dt \geq Cb^{\beta/2},$$

and thus, using the convexity of  $x \mapsto x^p$ ,

$$\int_0^1 |B_t|^p dt \geq Cb^{\beta p/2}.$$

Using (3.10), this achieves the proof of ii).

*Proof of iii).* For  $b$  small enough, we have:

$$\begin{aligned} B_t &= E_3(K_{t,b}(X_1)) - f_3(t) \\ &= E(f_3(\xi)) - f_3(t) \\ &= 2(E(\xi) - t) \\ &\geq Cb(1 - 2t), \end{aligned}$$

where  $\xi$  is defined as in Lemma 4 and last inequality is a consequence of (3.6).

As  $|1 - 2t|^p$  is integrable on  $[0, 1]$ , it follows:

$$\int_0^1 |B_t|^p dt \geq Cb^p.$$

Using Equation (3.10) we deduce Equation (3.5).

### 3.6 Proof of Lemma 6

Since  $x \mapsto |x|^p$  is a convex function

$$|B_t|^p = \left| E \left( \hat{f}_b(t) - f(t) \right) \right|^p \leq E \left( \left| \hat{f}_b(t) - f(t) \right|^p \right),$$

which implies (3.10) taking the integral on  $[0, 1]$  in both sides of the last inequality.

Since  $\|Z_t\|_p \leq \|B_t\|_p + \|\hat{f}_b - f\|_p$ , we deduce, using (3.10) that, for all function  $f$ ,

$$E \left( \|Z_t\|_p^p \right) \leq 2^{p-1} E \left( \|B_t\|_p^p + \|\hat{f}_b - f\|_p^p \right) \leq 2^p E \left( \|\hat{f}_b - f\|_p^p \right).$$

Then, we have

$$\begin{aligned} E \left( \|\hat{f}_b - f\|_p^p \right) &\geq 2^{-p} E \left( \|Z_t\|_p^p \right) \\ &= 2^{-p} E \left( \int_0^1 |Z_t|^p dt \right) \\ &= 2^{-p} \int_0^1 E \left( |Z_t|^p \right) dt \\ &\geq 2^{-p} \int_0^1 \left( E \left( Z_t^2 \right) \right)^{\frac{p}{2}} dt. \end{aligned}$$

This proves (3.11).

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